PNU-NTG-02/2002

A test of the instanton vacuum with low-energy theorems of the axial anomaly

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(Dated: July 2003)

Abstract

We revisit the QCD+QED axial anomaly low-energy theorems which give an exact relation between the matrix elements of the gluon and photon parts of the axial anomaly operator equation within the framework of the *effective action* derived from the instanton vacuum. The matrix elements between the vacuum and two photon states and between the vacuum and two gluon states are investigated for arbitrary N_f in the chiral limit. Having gauged the effective action properly, we show that the model does exactly satisfy the low-energy theorems.

Keywords: Axial anomaly, Low energy theorems, Instanton vacuum, effective action, nonlocal chiral quark model.

PACS numbers: 11.30.Rd, 12.38.Lg

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1. The quantum fluctuations may destroy the symmetries of the initial classical Lagrangian. One of the most important examples is the famous axial anomaly in gauge theories. The axial anomaly leads to many interesting nonperturbative features intimately related to the topologically nontrivial structure of the vacuum.

The axial anomaly in the divergence of the singlet axial-vector current in QCD + QED brings forth a low-energy theorem for the matrix elements of this operator equation between the vacuum and two–photon states (see, for example, a review [1]):

$$\left\langle 0 \left| N_f \frac{g^2}{32\pi^2} G\tilde{G} \right| 2\gamma \right\rangle = \frac{N_c}{8\pi^2} \sum_f e_f^2 F^{(1)} \tilde{F}^{(2)} \tag{1}$$

at $q^2 \ll m_{\eta'}^2$ ($m_{\eta'}$ is the η'-meson mass around 1 GeV). N_f is the number of flavors, g is the QCD coupling constant with $G\tilde{G} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} G^a_{\mu\nu} G^a_{\lambda\sigma}$ and N_c is the number of colors. Here, $G^a_{\mu\nu}$ stands for the operator of the gluon field strengths. e_f represent the electric charges of quarks. The photon part of Eq.(1) can be expressed explicitly as $F^{(i)}_{\mu\nu} = \epsilon^{(i)}_\mu q_{i\nu} - \epsilon^{(i)}_\nu q_{i\mu}$, where $\epsilon^{(1,2)}_\mu$ and $q_{1,2}$ are, respectively, polarizations and momenta of photons, and $q = q_1 + q_2$. In perturbation theory, it leads to at least results of order $e_f^2 g^4$ for the left-hand side of Eq. (1).

Another nontrivial low-energy theorem concerns the matrix element between the vacuum and two-gluon states:

$$\langle 0 | g^2 G\tilde{G} | 2\text{gluons} \rangle = 0 \tag{2}$$

at the same limit as in Eq.(1). Hence, the solution of these theorems given by Eqs.(1,2) is only pertinent to the nonperturbative phenomena related to the topologically nontrivial structure of the QCD vacuum.

Without any doubt instantons represent a very important topologically nontrivial component of the QCD vacuum. The most important parameters of the QCD instanton vacuum are the average instanton size ρ and inter-instanton distance R. Shuryak [2, 3] estimated these two parameters phenomenologically as $\rho \simeq 1/3\,\text{fm}$, $R \simeq 1\,\text{fm}$. It was confirmed by theoretical variational calculations [4, 5] and recent lattice simulations of the QCD vacuum [6]. In particular, the spontaneous breakdown of chiral symmetry is realized very well via the instanton liquid model. Hence, instantons play a pivotal and more significant role in describing the lightest hadrons and their interactions, compared with confinement forces [7, 8].

Some years ago, one of the authors showed that the low-energy QCD effective chiral action from the instanton vacuum [5, 9] satisfies the low energy theorems with an accuracy of about 17 % [10, 11]. This discrepancy is due to the fact that the vector and axial-vector currents are not conserved because of the momentum-dependent dynamical quark mass in the effective action [12, 13, 14, 15]. The nonconservation of the Nöther currents breaks various Ward identities in the model. In order to remedy these problems, we shall gauge the low-energy QCD effective partition function properly, from which the conserved currents can be derived.

In the present paper, we shall continue to test the low-energy QCD effective chiral action from the instanton vacuum in the chiral limit aiming at resolving the discrepancy existing in Ref. [10]. Having gauged the low-energy QCD effective partition function, we shall show that it does exactly satisfy these theorems as it should be.

2. We have to calculate the quark determinant in the presence of instantons and external electromagnetic field v_μ, and then average it over the collective coordinates of the instantons to find the effective low-energy QCD partition function. The first thing we have to do is to find the extended zero-modes $\tilde{\Phi}_0$ for the quark placed into the instanton field A^I_μ and the electromagnetic field v_μ, which obey the following equation:

$$(i\not{\partial}_x + g\not{A}^I(x) + e\not{v}(x))\tilde{\Phi}_0(x) = 0 \tag{3}$$

The extended zero-modes satisfy the U(1) gauge transformation

$$\tilde{\Phi}_0 = L\Phi'_0, \;\; L(x) = \exp\left(iex_\mu \int_0^1 v_\mu(\alpha x)d\alpha\right), \tag{4}$$

where L is simply a path-ordered exponent $P\exp(i\int_0^x v_\mu(s)ds_\mu)$ with a straight path connecting the points 0 to x [16, 17]. Then Eq. (3) is reduced to

$$(i\not{\partial}_x + g\not{A}^I(x) + e\not{v}'(x))\Phi'_0(x) = 0, \tag{5}$$

where

$$v'_\mu(x) = x_\rho \int_0^1 F_{\rho\mu}(\alpha x)\alpha d\alpha \tag{6}$$

has an explicit gauge-invariant form. An approximate solution of Eq. (5) is

$$\Phi'_0 = \Phi_0 - S^I e\not{v}'\Phi_0, \tag{7}$$

which can be expanded to any desired order of ev'_μ. Here, $S^I = (i\not{\partial} + g\not{A}^I + im)^{-1}$ is a quark propagator (with a small mass m) in the field of a single instanton:

$$S^I = \frac{|\Phi_0\rangle\langle\Phi_0|}{im} + S_{NZ}, \quad S_{NZ} = \sum_{n\neq 0}\frac{|\Phi_n\rangle\langle\Phi_n|}{\lambda_n + im} \tag{8}$$

where Φ_n is defined by $(i\not{\partial} + g\not{A}^I)|\Phi_n\rangle = \lambda_n|\Phi_n\rangle$ (with the convention $\lambda_0 = 0$). The quark propagator of the non-zero mode S_{NZ} in the single instanton field was derived in an exactly closed form [18]. It is clear from this formula that the quark propagator is reduced to the free one at a short distance as well as a long distance. We follow here the argument by Ref. [5] and will assume the following relation:

$$S_{NZ}(x,y) \simeq S_0(x-y). \tag{9}$$

Combining Eq.(9) with Eqs.(4,5,7), we obtain the extended zero mode $\tilde{\Phi}_0 = L\Phi'_0$. Now, we consider the quark propagator $\tilde{S}$ in the presence of the single instanton A^I_μ and v_μ with the assumption given by Eq.(9), so that we obtain for the quark propagator $\tilde{S}^I$:

$$\begin{aligned}\tilde{S}^I &= (i\not{\partial} + g\not{A}^I + e\not{v} + im)^{-1} = S^I - S^I e\not{v} S^I + \cdots \\ &= \frac{|\Phi_0\rangle\langle\Phi_0|}{im} + S_{NZ} - \left(\frac{|\Phi_0\rangle\langle\Phi_0|}{im} + S_{NZ}\right) e\hat{v} \left(\frac{|\Phi_0\rangle\langle\Phi_0|}{im} + S_{NZ}\right) + \cdots \\ &\simeq \tilde{S}_0 + \frac{|\tilde{\Phi}_0\rangle\langle\tilde{\Phi}_0|}{im},\end{aligned} \tag{10}$$

where $\tilde{S}_0 = (i\not{\partial} + e\not{v} + im)^{-1}$.

Now, we are in a position to consider the ensemble of instantons $A_\mu = \sum_I A_{I\mu}$, representing the instanton-liquid picture of the QCD vacuum. This sum consists of both instantons and antiinstantons. The quarks are moving in the presence of the instanton ensemble A_μ and external electromagnetic field v. Thus, we can derive an extended zero-mode approximation for the quark determinant.

The quark propagator $\tilde{S}$ can be expanded with respect to a single instanton:

$$\begin{aligned}\tilde{S} &= \tilde{S}_0 - \sum_I \tilde{S}_0 g\not{A}_I \tilde{S}_0 + \sum_{I,J} \tilde{S}_0 g\not{A}_I \tilde{S}_0 g\not{A}_J \tilde{S}_0 + \ldots &(11)\\ &= \tilde{S}_0 + \sum_I (\tilde{S}_I - \tilde{S}_0) + \sum_{I\neq J} (\tilde{S}_I - \tilde{S}_0)\tilde{S}_0^{-1}(\tilde{S}_J - \tilde{S}_0) &(12)\\ &\quad + \sum_{I\neq J,\, J\neq K} (\tilde{S}_I - \tilde{S}_0)\tilde{S}_0^{-1}(\tilde{S}_J - \tilde{S}_0)\tilde{S}_0^{-1}(\tilde{S}_K - \tilde{S}_0) + \ldots\ .\end{aligned}$$

Here, the indices $I, J, K, \cdots$ designate both instantons and anti-instantons. In Eq.(12), we have resummed the contributions corresponding to an individual instanton. $\tilde{S}_I$ stands for the quark propagator in the fields of the single instanton I as well as in the external electromagnetic field v and represents the sum of the zero and non-zero modes. At a large distance from the center of the instanton, $\tilde{S}_I$ becomes the propagator $\tilde{S}_0$ in the presence of the external field v. Equation (12) is obviously an extended quark propagator of that from Ref. [5] (see also review [19]). Thus, Eq. (10) can be written as

$$\left(\tilde{S}_I - \tilde{S}_0\right)(x,y) \simeq \frac{\tilde{\Phi}_{I,0}(x)\tilde{\Phi}^\dagger_{I,0}(y)}{im}. \tag{13}$$

In the present case, the expansion given by Eq.(12) becomes

$$\begin{aligned}\tilde{S}(x,y) &\simeq \tilde{S}_0(x,y) + \sum_I \frac{\tilde{\Phi}_{I,0}(x)\tilde{\Phi}^\dagger_{I,0}(y)}{im} &(14)\\ &\quad + \sum_{I\neq J} \frac{\tilde{\Phi}_{I,0}(x)}{im}\left(\int d^4r\, \tilde{\Phi}^\dagger_{I,0}(r)(i\not{\partial} + e\not{v} + im)\tilde{\Phi}_{J,0}(r)\right)\frac{\tilde{\Phi}^\dagger_{J,0}(y)}{im} + \cdots .\end{aligned}$$

Defining the following overlapping integrals

$$\tilde{a}_{IJ} = -\int d^4r\, \tilde{\Phi}^\dagger_{I,0}(r)(i\not{\partial} + e\not{v})\tilde{\Phi}_{J,0}(r), \tag{15}$$

we are led to the expression:

$$\tilde{S}(x,y) \simeq \tilde{S}_0(x,y) + \sum_{I,J} \tilde{\Phi}_{I,0}(x)\left(\frac{1}{\tilde{a} + im}\right)_{IJ} \tilde{\Phi}^\dagger_{J,0}(y). \tag{16}$$

The total quark determinant should be now splitted into two parts, *i.e.* low and high frequencies (with respect to a mass parameter $M_1 \sim \rho^{-1}$): $\mathrm{Det} = \mathrm{Det}_{\mathrm{low}} \times \mathrm{Det}_{\mathrm{high}}$ [5]. The high-frequency part $\mathrm{Det}_{\mathrm{high}}$ can be written as a product of the determinants in the field of individual instantons, while the low-frequency one $\mathrm{Det}_{\mathrm{low}}$ has to be treated approximately, would-be zero modes being taken into account only. Thus, we get for the low-frequency part of the quark determinant as follows:

$$\ln \mathrm{Det}_{\mathrm{low}} = \sum_f \int_{M_1}^{m_f} idm[\mathrm{Tr}(\tilde{S} - \tilde{S}_0) - \mathrm{Tr}(\tilde{S}_0 - S_0)], \tag{17}$$

where

$$\mathrm{Tr}(\tilde{S}-\tilde{S}_0)=\mathrm{Tr}\frac{1}{\tilde{a}+im}. \qquad (18)$$

The second term in Eq.(17) can be included in the normalization factor. Taking into account the external gauge field v, we get

$$\mathrm{Det}_{\mathrm{low}}=\det_{N,v}=\det\tilde{B},\quad \tilde{B}_{IJ}=\tilde{a}_{JI}+im\delta_{JI}. \qquad (19)$$

If we had turned off the external field v, we would have obtained exactly the same quark determinant $\det_N$ by Lee and Bardeen [20]. Having fermionized the representation of Eq. (19) [10, 11], we derive the gauged quark determinant in terms of the constituent quarks ψ_f:

$$\begin{aligned}\det_{N,v} &= \int D\psi D\psi^\dagger \exp(\int d^4x\sum_f \psi_f^\dagger(i\partial\!\!\!/+e_f v\!\!\!/)\psi_f)\\ &\times \prod_f(\prod_+^{N_+}(im_f-\tilde{V}_+[\psi_f^\dagger,\psi_f])\prod_-^{N_-}(im_f-\tilde{V}_-[\psi_f^\dagger,\psi_f])),\end{aligned} \qquad (20)$$

where

$$\begin{aligned}\tilde{V}_\pm[\psi_f^\dagger,\psi_f] &= \int d^4x(\psi_f^\dagger(x)(i\partial\!\!\!/+e_f v\!\!\!/)\tilde{\Phi}_{\pm,0}(x;\xi_\pm))\int d^4y(\tilde{\Phi}_{\pm,0}^\dagger(y;\xi_\pm)(i\partial\!\!\!/+e_f v\!\!\!/\psi_f(y)) & (21)\\ &= \int d^4x(\psi_f^\dagger(x)L_f(x)i\partial\!\!\!/\Phi_{\pm,0}(x;\xi_\pm))\int d^4y(\Phi_{\pm,0}^\dagger(y;\xi_\pm)i\partial\!\!\!/L_f^\dagger(y)\psi_f(y)). & (22)\end{aligned}$$

Here, we have used Eqs.(7,9) to derive Eq.(22) from Eq.(21). The range of the integration in Eq.(21) is cut at ρ, since it is defined by zero-mode functions $\Phi_{\pm,0}$. In the present case, we assume the soft-photon external field $q\rho\ll 1$ (q is a photon momentum) which simplifies the gauge factor:

$$L_f(x)=\exp\left(ie_f(x-z)_\mu\int_0^1 v_\mu(z+\alpha(x-z))d\alpha\right)\simeq\exp\left(ie_f(x-z)_\mu v_\mu(z)\right) \qquad (23)$$

Note that external v_μ field gauges not only the kinetic term of the effective action but also its interaction term $\tilde{V}_\pm[\psi_f^\dagger,\psi_f]$ in Eq. (21). The reason is obvious: It is the nonlocal interaction induced by instantons. The external v_μ-field is present here due to the factor L_f attached to each fermionic line. This factor provides us an approximate gauge invariance of the interaction term $\tilde{V}_\pm[\psi_f^\dagger,\psi_f]$ under the gauge transformation which is valid in the soft-photon limit $q\rho\ll 1$. The remaining problem is to average the quark determinant over collective coordinates $\xi_\pm$. It is a rather simple procedure, since the low density of the instanton medium ($\pi^2\left(\frac{\rho}{R}\right)^4\sim 0.1$) allows us to average over positions and orientations of the instantons independently. In the following, we will assume the chiral limit. Thus, Eq.(20) gives us the gauged partition function:

$$Z_N[v]=\int D\psi D\psi^\dagger\exp\left(\int d^4x\sum_f\psi_f^\dagger(i\partial\!\!\!/+e_f v\!\!\!/)\psi_f\right)\tilde{W}_+^{N_+}\tilde{W}_-^{N_-}, \qquad (24)$$

where

$$\tilde{W}_\pm=\int d\xi_\pm\prod_f(\tilde{V}_\pm[\psi_f^\dagger\,\psi_f])=(i)^{N_f}\left(\frac{4\pi^2\rho^2}{N_c}\right)^{N_f}\int\frac{d^4z}{V}\det_f i\tilde{J}_\pm(z) \qquad (25)$$

and

$$\tilde{J}_{\pm}(z)_{fg} \;=\; \int \frac{d^4k d^4l}{(2\pi)^8} \exp(-i(k-l)z)\, F((k+e_f v(z))^2) F((l+e_g v(z))^2)\, \psi_f^\dagger(k) \frac{1}{2}(1\pm\gamma_5)\psi_g(l). \quad (26)$$

The form factor $F(k^2)$ is related to the zero-mode wave function in momentum space $\Phi_\pm(k;\xi_\pm)$ and turns out to be $F(k^2) = -t\frac{d}{dt}\left[I_0(t)K_0(t) - I_1(t)K_1(t)\right],\ t = \frac{1}{2}\sqrt{k^2}\rho$, where I_i and K_i are modified Bessel functions. Thus, we find that the gauged partition function in the soft-photon limit is nothing but the original partition function with the gauged form factor $F((k+e_f v(z))^2)$. Details of how to derive the effective action from the partition function are well described in Ref. [5]. The self-consistency condition of the exponentiation and bosonization at the common saddle-point leads to the following equation for the dynamical quark mass M:

$$4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2F^4(k)}{M^2F^4(k)+k^2} = \frac{N}{V}, \quad (27)$$

where N/V denotes the finite density of instantons and antiinstantons in Euclidean space.

In the quasiclassical (saddle-point) approximation, any gluon operator receives its main contribution from the instanton background. As an example, for one instanton (anti-instanton)$I(\bar{I})$, the following gluonic operators can be expressed in terms of the instanton background:

$$g^2G^2(x) = \frac{192\rho^4}{[\rho^2+(x-z)^2]^4} = f(x-z), \quad g^2G\tilde{G}(x) = \pm f(x-z). \quad (28)$$

The derivation of the correlators with the gluonic operator $G\tilde{G}$ comes down to that of the following partition function:

$$\hat{Z}_N[v,\kappa] = Z_N^{-1} \int D\psi D\psi^\dagger \exp(-\hat{S}_{\text{eff}}), \quad (29)$$

where the effective action, $\hat{S}_{\text{eff}}$ in the presence of an external fields $v_\mu(x)$ and $\kappa(x)$ coupled to $G\tilde{G}$, is given by

$$\begin{aligned} -\hat{S}_{\text{eff}} \;=\; & \int d^4x \sum_f \psi_f^\dagger(i\partial\!\!\!/ + e_f v\!\!\!/)\psi_f + \left(\frac{2V}{N}\right)^{N_f-1} \int d^4z\, \det(iM_+(z)\tilde{J}_+(z)) \\ & + \left(\frac{2V}{N}\right)^{N_f-1} \int d^4z\, \det(iM_-(z)\tilde{J}_-(z)), \end{aligned} \quad (30)$$

where $M_\pm(z) = (1 \pm \int dx \kappa(x) f(x-z))^{(N_f-1)^{-1}}\, M$ [10]. Using the bozonization again, one can show that the effective action $\hat{S}_{\text{eff}}[\mathcal{M}_\pm, v, \kappa]$ turns out to be

$$\begin{aligned} -\hat{S}_{\text{eff}}[\mathcal{M}_\pm, v, \kappa] \;=\; & \int d^4z \left(-(N_f-1)\left(\frac{2V}{N}\right)^{-1} (\det \mathcal{M}_\pm)^{\frac{1}{N_f-1}}\right) \\ & + \operatorname{Tr}\ln\Big[P\!\!\!/ + iMF(P^2)\mathcal{M}_+F(P^2)\,(1+(\kappa f))^{N_f^{-1}}\frac{1}{2}(1+\gamma_5) \\ & + iMF(P^2)\mathcal{M}_-F(P^2)\,(1-(\kappa f))^{N_f^{-1}}\frac{1}{2}(1-\gamma_5)\Big] \end{aligned} \quad (31)$$

in the presence of the external fields v_μ and κ. Here, P_μ denotes the covariant momentum $P_\mu := p_\mu + eQv_\mu$ with $eQ_{fg} = e_f\delta_{fg}$. The matrices $\mathcal{M}_\pm$ describe the meson fields and $(\kappa f) = \int d^4x\kappa(x)f(x-z)$.

Since the matrix element between the vacuum and two-photon state is defined by the functional derivative $\left.\frac{\delta \hat{Z}_N[\kappa,v]}{\delta\kappa(x)\delta v_\mu(x_1)\delta v_\nu(x_2)}\right|_{\kappa,v=0}$, we need to keep only the terms of order $\mathcal{O}(\kappa)$ and $\mathcal{O}(v^2)$ in $\hat{S}_{\rm eff}[\mathcal{M}_\pm = 1, v, \kappa]$ defined by Eq.(31).

We will employ for the explicit calculation the Schwinger method [21], which was developed for quantum electrodynamics and later extended to QCD [22], in order to expand $\hat{S}_{\rm eff}$ with respect to derivatives of the photon background field. Setting $\mathcal{M}_\pm = 1$, considering the first order of κ, and defining $M(P) = MF^2(P)$, we get

$$-\hat{S}_{\rm eff}[\mathcal{M}_\pm = 1, v, \kappa] = {\rm Tr}\ln[\not{P} + iM(P)] + \frac{i}{N_f}{\rm Tr}\left[(\not{P} + iM(P))^{-1}M(P)(\kappa f)\gamma_5\right], \tag{32}$$

where ${\rm Tr} = {\rm tr}_{cf\gamma}\int d^4x\langle x|.....|x\rangle$ and ${\rm tr}_{cf\gamma}$ denotes the trace over color, flavor, and Dirac spaces. The quark propagator with the covariant derivative can be treated as follows:

$$(\not{P} + iM(P))^{-1} = \left(P^2 + M^2(P) + \frac{eQ}{2}\sigma\cdot F + i[\not{P}, M(P)]\right)^{-1}(\not{P} - iM(P)), \tag{33}$$

where $\sigma\cdot F = \sigma_{\mu\nu}F_{\mu\nu}$. We need at least four γ matrices with one γ_5 so that we may obtain a nonvanishing result with the trace over Dirac space (${\rm tr}_\gamma$). Thus, the relevant nonvanishing part of the effective action is:

$$\begin{aligned} -S_{kf} \;=\; & \frac{1}{N_f}{\rm Tr}\left\{\left[(P^2 + M^2(P))^{-1}\left(\frac{eQ}{2}\sigma\cdot F + i[\not{P}, M(P)]\right)\right]^2\right. \\ \times\; & \left.(P^2 + M^2(P))^{-1}(M^2(P) + i\not{P}M(P))(\kappa f)\gamma_5\right\}. \end{aligned} \tag{34}$$

We observe that $[\not{P}, M(P)]$ is of order $\mathcal{O}(e)$ and M is a function of P^2. Then

$$\begin{aligned} -S_{kf} \;=\; & \frac{1}{N_f}{\rm Tr}\left[(p^2 + M^2(p^2))^{-1}\left(\frac{eQ}{2}\sigma\cdot F\right)^2(p^2 + M^2(p^2))^{-1}M^2(p^2)(\kappa f)\gamma_5\right. \\ -\; & \left\{(p^2 + M^2(p^2))^{-1}\frac{eQ}{2}\sigma\cdot F,\; (p^2 + M^2(p^2))^{-1}[\not{P}, M(P^2)]\right\} \\ \times\; & \left.(p^2 + M^2(p^2))^{-1}\not{p}M(p^2)(kf)\gamma_5\right] \;+\; \mathcal{O}(e^3). \end{aligned} \tag{35}$$

Here, we have taken into account the term to order $\mathcal{O}(e^2)$ only. Hence, what we need is to calculate $[\not{P}, M(P^2)]$, which becomes in the gauge $[p_\mu, v_\mu] = 0$:

$$[\not{P}, M(P^2)] = -2ieQF_{\mu\nu}\gamma_\nu p_\mu\frac{dM(p^2)}{dp^2} + O(e^2, \partial^2). \tag{36}$$

Putting them together, we finally arrive at the expression:

$$-S_{kf} = \int\frac{d^4p}{(2\pi)^4}\frac{M^2(p^2) - p^2\frac{dM^2(p^2)}{dp^2}}{(p^2 + M^2(p^2))^3}{\rm tr}\left(\frac{eQ}{2}\sigma\cdot F\right)^2(kf)\gamma_5. \tag{37}$$

It was shown in the previous paper [10] that if we neglect the momentum dependence of the M, then we exactly reproduce the low-energy theorem in Eq.(1). Hence, In order to prove that the gauged effective action exactly satisfies the low-energy theorem of the axial anomaly, it is enough to show that the following ratio $R = \frac{J_1}{J_2}$ with

$$J_1 = \int p^2 dp^2 \frac{M^2(p^2) - p^2 \frac{dM^2(p^2)}{dp^2}}{(p^2 + M^2(p^2))^3}, \quad J_2 = \int dp^2 p^2 \frac{M^2(0)}{(p^2 + M^2(0))^3} \tag{38}$$

becomes unity. Replacing the variable [14] by $s = \frac{M^2(p^2)}{p^2}$, we immediately have the ratio $R = 1$. As a result, the nonlocal chiral quark model from the instanton vacuum exactly satisfies the low-energy theorem given by Eq.(1).

Now, we briefly present some calculations related to Eq.(2). This matrix element can be written in the form:

$$\begin{aligned} &\langle 0|g^2 G\tilde{G}|g(\epsilon^{(1)}, q_1), g(\epsilon^{(2)}, q_2)\rangle \\ &= \epsilon^{(1)a_1}_{\mu_1} \epsilon^{(2)a_2}_{\mu_2} \int \partial_2^2 \, \partial_1^2 \, \langle 0|T g^2 G\tilde{G} A^{a_1}_{\mu_1}(x_1) A^{a_2}_{\mu_2}(x_2)|0\rangle \exp i(q_1 x_1 + q_2 x_2) dx_1 dx_2, \end{aligned} \tag{39}$$

where $A^a_\mu(x)$ denotes a total gluon field. $\epsilon^{(i)a_i}_{\mu_i}$ and q_i are the polarization vectors and the momentum of gluons, respectively.

As usual, we expand the total field $A^a_\mu(x)$ around the instanton background. The main term in Eq.(39) is the contribution of the instanton background and is of order $\mathcal{O}(g^{-2})$. The next term is due to the perturbative fluctuations over the instanton background and corresponds to the contribution of order $\mathcal{O}(g^2)$. It is easy to see from previous considerations that the term to order $\mathcal{O}(g^{-2})$ is given by the following expression:

$$Z_N^{-1} \int D\psi D\psi^\dagger \exp(-S_{\rm eff}) \left(Y_{G\tilde{G}AA+}(x) + Y_{G\tilde{G}AA-}(x) \right) \tag{40}$$

with

$$\begin{aligned} Y_{G\tilde{G}AA\pm} &= \pm \left(\frac{2V}{N}\right)^{N_f - 1} (iM)^{N_f} \int d^4 z \, f(x - z) \\ &\times \int dO (-\partial_1^2) A^{I(\bar{I})a_1}_{\mu_1}(x_1)(-\partial_2^2) A^{I(\bar{I})a_2}_{\mu_2}(x_2) \det J_\pm(z), \end{aligned} \tag{41}$$

where the instanton(anti-instanton) is located at the point z with its orientation O [10] over which we integrate.

Repeating the bosonization, we obtain the result for the $\mathcal{O}(g^{-2})$ contribution which is proportional to $\mathrm{Tr}[(i\partial\!\!\!/ + iMF^2)^{-1} iMF^2 \gamma_5]$. Hence, one can easily show that the contribution from the term to order $\mathcal{O}(g^{-2})$ vanishes.

The contribution from the next order ($\mathcal{O}(g^2)$) comes from two different diagrams. The first diagram is the direct contribution of the operator $g^2 G\tilde{G}$ which is equal to $-g^2 G^{(1)}\tilde{G}^{(2)}$, where $2G^{(1)}\tilde{G}^{(2)} = \epsilon^{\mu\nu\lambda\sigma} G^{(1)a}_{\mu\nu} G^{(2)a}_{\lambda\sigma}$, $G^{(i)a}_{\mu\nu} = \epsilon^{(i)}_\mu q_{i\nu} - \epsilon^{(i)}_\nu q_{i\mu}$. The factors at the vertices of the second one are $g\gamma_\mu \lambda_a/2$ and $iMfF^2\gamma_5 N_f^{-1}$.

A comparison with previous calculations ends up with the result that the contribution from the second loop–diagram is equal in magnitude but opposite in sign to that from the first one at $q^2 \to 0$. Because of this cancellation, the contribution from order $\mathcal{O}(g^2)$ vanishes in the limit $q^2 \to 0$. Hence, we arrive exactly at the same conclusion as in the previous case.

From the above calculations, we conclude that the nonlocal chiral quark model from the instanton vacuum does exactly satisfy the low-energy theorems, once we have properly gauged the low-energy QCD partition function from the instanton vacuum.

3. The solution of the low-energy theorems of the axial anomaly is nontrivially related to the nonperturbative instanton structure of the QCD vacuum. The effective action approach based on the instanton vacuum exactly satisfies these low-energy theorems.

This approach provides a solid ground for future investigations into different amplitudes for the nonperturbative conversion of gluons into hadrons and photons. Further studies relevant to all of these problems are under way.

Acknowledgments

One of the authors(MM) acknowledges the warm hospitality of the Nuclear Theory Group of PNU, where this work was initiated. The present work is supported by the Korean Research Foundation (KRF-2002-041-C00067). The work of MM is supported in part by INTAS, BMBF and SNF grants.

[1] M. A. Shifman, Phys. Rept. **209** (1991) 341 [Sov. Phys. Usp. **32** (1989 UFNAA,157,561-598.1989) 289].

[2] E. V. Shuryak, Nucl. Phys. B **203** (1982) 93.

[3] E. V. Shuryak, Nucl. Phys. B **203** (1982) 116.

[4] D. Diakonov and V. Y. Petrov, Nucl. Phys. B **245** (1984) 259.

[5] D. Diakonov and V. Y. Petrov, Nucl. Phys. B **272** (1986) 457.

[6] T. DeGrand, Phys. Rev. D **64** (2001) 094508 [arXiv:hep-lat/0106001].

[7] D. Diakonov, arXiv:hep-ph/0205054.

[8] M. C. Chu, J. M. Grandy, S. Huang and J. W. Negele, Phys. Rev. D **49** (1994) 6039 [arXiv:hep-lat/9312071].

[9] D. Diakonov, M. V. Polyakov and C. Weiss, Nucl. Phys. B **461** (1996) 539 [arXiv:hep-ph/9510232].

[10] M. M. Musakhanov and F. C. Khanna, Phys. Lett. B **395** (1997) 298 [arXiv:hep-ph/9610418].

[11] E. D. Salvo and M. M. Musakhanov, Euro. Phys. J. C **5** (1998) 501 [arXiv:hep-ph/9706537].

[12] R. D. Ball and G. Ripka, arXiv:hep-ph/9312260.

[13] R. D. Bowler and M. C. Birse, Nucl. Phys. A **582** (1995) 655 [arXiv:hep-ph/9407336].

[14] R. S. Plant and M. C. Birse, Nucl. Phys. A **628** (1998) 607 [arXiv:hep-ph/9705372].

[15] W. Broniowski, arXiv:hep-ph/9909438.

[16] M. Chretien and R.E. Peierls, Proc. R. Soc. London A **223**, 468 (1954)

[17] A. V. Radyushkin, arXiv:hep-ph/0101227.

[18] L. S. Brown, R. D. Carlitz, D. B. Creamer, C. K. Lee, *Phys. Rev.* **D 17**, 1583 (1978).

[19] T. Schafer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998) [arXiv:hep-ph/9610451].

[20] C. k. Lee and W. A. Bardeen, Nucl. Phys. B **153** (1979) 210.

[21] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[22] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov and M. A. Shifman, Sov. J. Nucl. Phys. **39**, 77 (1984) [Yad. Fiz. **39**, 124 (1984)].